

# Primitivity of finitely presented monomial algebras

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## Abstract

We study prime monomial algebras. Our main result is that a prime finitely presented monomial algebra is either primitive or it has GK dimension one and satisfies a polynomial identity. More generally, we show this result holds for the class of *automaton algebras*; that is, monomial algebras that have a basis consisting of the set of words recognized by some finite state automaton. This proves a special case of a conjecture of the first author and Agata Smoktunowicz.

## 1 Introduction

We consider prime monomial algebras over a field  $k$ . Given a field  $k$ , a finitely generated  $k$ -algebra  $A$  is called a *monomial algebra* if

$$A \cong k\{x_1, \dots, x_d\}/I$$

for some ideal  $I$  generated by monomials in  $x_1, \dots, x_d$ . Monomial algebras are useful for many reasons. First, Gröbner bases associate a monomial algebra to a finitely generated algebra, and for this reason monomial algebras can be used to answer questions about ideal membership and the Hilbert series for general algebras. Also, many difficult questions for algebras reduce to combinatorial problems for monomial algebras and can be studied in

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terms of forbidden subwords. Monomial algebras are consequently a rich area of study. The paper of Belov, Borisenko, and Latyshev [4] is an interesting survey of what is known about monomial algebras.

The first author and Smoktunowicz [3] studied prime monomial algebras of quadratic growth, showing that they are either primitive or have nonzero Jacobson radical. By the Jacobson density theorem, primitive algebras are dense subrings of endomorphism rings of a module over a division algebra. For this reason, primitive ideals are an important object of study and their study is often an important intermediate step in classifying finite dimensional representations of an algebra. The first author and Smoktunowicz [3] also made the following more general conjecture about monomial algebras.

**Conjecture 1.1.** *Let  $A$  be a prime monomial algebra over a field  $k$ . Then  $A$  is either PI, primitive, or has nonzero Jacobson radical.*

These types of “trichotomies” are abundant in ring theory and there are many examples of algebras for which either such a trichotomy is known to hold or is conjectured; e.g., just infinite algebras over an uncountable field [6], Small’s conjecture [2, Question 3.2]: a finitely generated prime Noetherian algebra of quadratic growth is either primitive or PI.

We prove Conjecture 1.1 in the case that  $A$  is finitely presented; in fact we are able to prove it more generally when  $A$  is a prime *automaton algebra*; that is, when  $A$  is a monomial algebra with a basis given precisely by the words recognized by a finite state machine.

Our main result is the following theorem.

**Theorem 1.2.** *Let  $k$  be a field and let  $A$  be a prime finitely presented monomial  $k$ -algebra. Then  $A$  is either primitive or  $A$  satisfies a polynomial identity.*

A consequence of this theorem is that any finitely generated prime monomial ideal  $P$  in a finitely generated free algebra  $A$  is necessarily primitive, unless  $A/P$  has GK dimension at most 1.

We prove our main result by showing that a finitely presented monomial algebra  $A$  has a well-behaved free subalgebra. In this case, well-behaved means that there the poset of left ideals of the subalgebra embeds in the poset of left ideals of  $A$  and nonzero ideals in our algebra  $A$  intersect the subalgebra non-trivially. A free algebra is primitive if it is free on at least two generators; otherwise it is PI. From this fact and the fact that  $A$  has a well-behaved free subalgebra, we are able to deduce that  $A$  is either primitive or PI.

In Section 2 we give some background on finite state automata and automaton algebras. In Section 3 we give some useful facts about primitive algebras and PI algebras that we use in obtaining our dichotomy. In Section 4 we prove our main result.

## 2 Automata theory and automaton algebras

In this section we give some basic background about finite state automata and automaton algebras. A finite state automaton is a machine that accepts as input words on a finite alphabet  $\Sigma$  and has a finite number of possible outputs. We give a more formal definition.

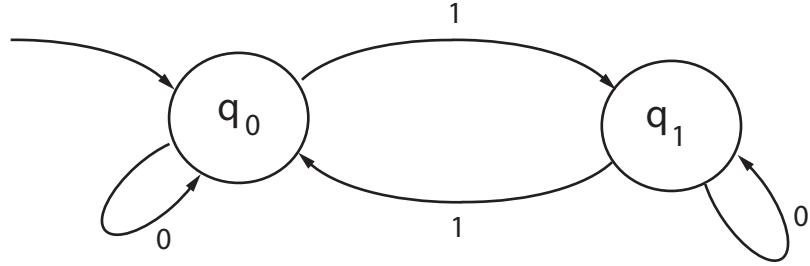


Figure 1: A finite state automaton with two states.

**Definition.** A *finite state automaton*  $\Gamma$  is a 5-tuple  $(Q, \Sigma, \delta, q_0, F)$ , where:

1.  $Q$  is a finite set of states;
2.  $\Sigma$  is a finite alphabet;
3.  $\delta : Q \times \Sigma \rightarrow Q$  is a transition function;
4.  $q_0 \in Q$  is the initial state;
5.  $F \subseteq Q$  is the set of accepting states.

We refer the reader to Sipser [14, page 35] for more background on automata. We note that we can inductively extend the transition function  $\delta$  to a function from  $Q \times \Sigma^*$  to  $Q$ , where  $\Sigma^*$  denotes the collection of finite words of  $\Sigma$ .<sup>1</sup>

We give an example.

**Example.** The finite state machine described in Figure 1 has alphabet  $\Sigma = \{0, 1\}$ , states  $\{q_0, q_1\}$  accepting state  $\{q_0\}$  and transition rules  $\delta(q_i, 1) = q_{1-i}$ ,  $\delta(q_i, 0) = q_i$  for  $i = 0, 1$ . In particular, if  $w$  is a word on  $\{0, 1\}$  then  $\delta(q_0, w)$  is  $q_0$  if and only if the number of ones in  $w$  is even.

**Definition.** Let  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  be a finite state automaton. We say that a word  $w \in \Sigma^*$  is *accepted* by  $\Gamma$  if  $\delta(q_0, w) \in F$ ; otherwise, we say  $w$  is *rejected* by  $\Gamma$ .

We now describe the connection between monomial algebras and finite state automata.

**Definition.** Let  $k$  be a field and let  $A = k\{x_1, x_2, \dots, x_n\}/I$  be a monomial algebra. We say that  $A$  is an *automaton algebra* if there exists a finite state automaton  $\Gamma$  with alphabet  $\Sigma = \{x_1, \dots, x_n\}$  such that the words  $w$  is accepted by  $\Gamma$  if and only if  $w \notin I$ .

**Remark 1.** Any finitely presented monomial algebra is an automaton algebra.

<sup>1</sup>We simply define  $\delta(q, \varepsilon) = q$  if  $\varepsilon$  is the empty word and if we have defined  $\delta(q, w)$  and  $x \in \Sigma$ , we define  $\delta(q, wx) := \delta(\delta(q, w), x)$ .

**Proof.** See Belov, Borisenko, and Latyshev [4, Proposition 5.4 p. 3528].  $\square$

We note that in order for finite state automaton  $\Gamma$  to give rise to a monomial algebra, the collection of words that are rejected by  $\Gamma$  must generate a two-sided ideal that does not contain any two sided words. This need not occur in general (see, for example, Figure 1, in which we take  $F = \{q_0\}$ ).

In general, an automaton algebra can have many different corresponding finite state automata. We may assume, however, that the corresponding finite state automaton is *minimal*.

**Definition.** We say that a finite state automaton  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  is *minimal* if for  $q_1, q_2 \in Q$  with  $q_1 \neq q_2$  we have

$$\{w \in \Sigma^* : \delta(q_1, w) \in F\} \neq \{w \in \Sigma^* : \delta(q_2, w) \in F\}$$

and for every  $q \in Q$  there is a word  $w \in \Sigma^*$  such that  $\delta(q_0, w) = q$ .

We note that this definition of minimality is slightly different from other definitions that appear in the literature. It can, however, be shown to be equivalent [16]. By the Myhill-Nerode theorem [8], if  $A$  is an automaton algebra, there is a minimal automaton  $\Gamma$  such that the images in  $A$  of the words accepted by  $\Gamma$  form a basis for  $A$ ; moreover, this automaton  $\Gamma$  is unique up to isomorphism. For this reason, we will often speak of *the* minimal automaton corresponding to an automaton algebra.

### 3 Primitivity and Polynomial identities

In this section we give some important background on primitive algebras and algebras satisfying a polynomial identity. We first recall the definitions of the two main concepts that make up our dichotomy.

**Definition.** A ring  $R$  is *left-primitive* if it has a faithful simple left  $R$ -module  $M$ .

Right-primitivity is defined analogously. Left-primitivity and right-primitivity often coincide; nevertheless there are examples of algebras which are left- but not right-primitive [5]. For the purposes of this paper, we will say that an algebra is *primitive* if it is *both* left- and right-primitive.

Two useful criteria for being primitive are given in the following remark.

**Remark 2.** Let  $R$  be a ring with unit. The following are equivalent:

1.  $R$  is left primitive;
2.  $R$  has a maximal left ideal  $I$  that does not contain a nonzero two sided ideal of  $R$ ;
3.  $R$  has a left ideal  $I$  such that  $I + P = R$  for every nonzero prime ideal  $P$ .

**Proof.** For the equivalence of (1) and (2), see Rowen [13, Page 152]; for the equivalence of (2) and (3) note that (2) trivially implies (3), and if  $I$  is a left ideal that does not contain a nonzero two-sided ideal of  $R$  then by Zorn's lemma we can find a maximal left ideal with this property.  $\square$

The other concept used in the dichotomy in the statement of Theorem 1.2 is that of being *PI*.

**Definition.** We say that a  $k$ -algebra  $A$  satisfies a *polynomial identity* if there is a nonzero noncommutative polynomial  $p(x_1, \dots, x_n) \in k\{x_1, \dots, x_n\}$  such that  $p(a_1, \dots, a_n) = 0$  for all  $(a_1, \dots, a_n) \in A^n$ . If an algebra  $A$  satisfies a polynomial identity we will say that  $A$  is *PI*.

Polynomial identity algebras are a natural generalization of commutative algebras, which, by definition, satisfy the polynomial identity  $xy - yx = 0$ . An important theorem of Kaplansky [7, p. 157, 6.3.1] shows that an algebra that is *both* primitive and PI is a matrix ring over a division algebra that is finite dimensional over its centre. Kaplansky's theorem shows that being primitive and being PI are in some sense incongruous and this incongruity is expressed in the fact that for many classes of algebras there are either theorems or conjectured dichotomies which state that the algebra must be either primitive or PI [6, 3, 2].

To prove Theorem 1.2, we rely on a reduction to free algebras. A free algebra on 1 generator is just a polynomial ring and hence PI. A free algebra on two or more generators is necessarily primitive. This result is due to Samuel [9, page 36]. We outline a proof of this fact, since Jacobson only describes the result for free algebras on precisely two generators.

**Theorem 3.1.** *A free algebra that is either countably infinitely generated or is generated by  $d$  elements for some natural number  $d \geq 2$  is primitive.*

**Proof.** Since a free algebra is isomorphic to its opposite ring, it is sufficient to prove left primitivity. First, if  $A$  is the free algebra on two generators, say  $A = k\{x, y\}$ , then we construct a left  $A$ -module  $M$  as follows. Let  $M = \sum_{i \geq 0} ke_i$  and let  $A$  act on  $M$  via the rules

$$xe_i = e_{i-1} \quad \text{and} \quad ye_i = e_{i^2+1},$$

where we take  $e_{-1} = 0$ . Then  $M$  is a faithful simple left  $A$ -module and so  $A$  is left primitive. Next observe that if  $A = k\{x, y\}$  then  $k + Ay$  is free on infinitely many generators  $y, xy, x^2y, \dots$  and hence a free algebra on a countably infinite number of generators is primitive [12]. It follows that if  $d \geq 2$  is a natural number then  $A = k\{x_1, \dots, x_d\}$  is again primitive since  $k + Ax_d$  is free on a countably infinite number of generators [12].  $\square$

We use Theorem 3.1 to prove the primitivity of prime automaton algebras of GK dimension greater than 1. To do this, we use a result that allows one to show that an algebra with a sufficiently well-behaved primitive subalgebra is itself necessarily primitive; we make this more precise in Proposition 3.2, but before stating this proposition we require a definition.

**Definition.** Let  $B$  be a subring of a ring  $A$ . We say that  $A$  is *nearly free* as a left  $B$ -module if there exists some set  $E = \{x_\alpha \mid \alpha \in S\} \subseteq A$  such that:

1.  $x_{\alpha_0} = 1$  for some  $\alpha_0 \in S$ ;
2.  $A = \sum_\alpha Bx_\alpha$ ;
3. if  $b_1, \dots, b_n \in B$  and  $b_1x_{\alpha_1} + \dots + b_nx_{\alpha_n} = 0$  then  $b_ix_{\alpha_i} = 0$  for  $i = 1, \dots, n$ .

We note that it is possible to be nearly free over a subalgebra without being free. For example, let  $A = \mathbb{C}[x]/(x^3)$  and let  $B$  be the subalgebra of  $A$  generated by the image of  $x^2$  in  $A$ . Then  $A$  is 3-dimensional as a  $\mathbb{C}$ -vector space while  $B$  is 2-dimensional. Hence  $A$  cannot be free as a left  $B$ -module. Let  $\bar{x}$  denote the image of  $x$  in  $A$ . Then

$$A = B + B\bar{x}.$$

Moreover  $A$  is  $\mathbb{N}$ -graded and  $B$  is the graded-subalgebra generated by homogeneous elements of even degree. Hence if  $b_1 + b_2\bar{x} = 0$  then  $b_1 = b_2\bar{x} = 0$ . Hence  $A$  is nearly free as a  $B$ -module.

**Proposition 3.2.** *Let  $A$  be a prime algebra and suppose that  $B$  is a right primitive subalgebra of  $A$  such that:*

1.  $A$  is nearly free as a left  $B$ -module;
2. every nonzero two-sided ideal  $I$  of  $A$  has the property that  $I \cap B$  is nonzero.

*Then  $A$  is right primitive.*

**Proof.** Pick a maximal right  $I$  of  $B$  that does not contain a nonzero two-sided ideal of  $B$ . Let  $E = \{x_\alpha : \alpha \in S\}$  be a subset of  $A$  satisfying:

1.  $A = \sum_{x_\alpha \in E} Bx_\alpha$ ;
2. if  $b_1x_{\alpha_1} + \dots + b_dx_{\alpha_d} = 0$ , then  $b_ix_{\alpha_i} = 0$  for every  $i$ ;
3.  $x_\beta = 1$  for some  $\beta \in S$ .

Then

$$IA = \sum_{\alpha \in S} Ix_\alpha.$$

We claim that  $IA$  is proper right ideal of  $A$ . If not then

$$1 = x_\beta = \sum a_k x_{\alpha_k},$$

for some  $a_k \in I$ . Since  $A$  is nearly free as a left  $B$ -module,  $x_\beta - ax_\beta = 0$  for some  $a \in I$ , contradicting the fact that  $I$  is proper. Thus  $IA$  is a proper right ideal. By Zorn's lemma, we can find a maximal right ideal  $L$  lying above  $IA$ . We claim that  $A/L$  is a faithful simple right  $A$ -module. To see this, suppose that  $L$  contains a nonzero prime ideal  $P$  of  $A$ . By assumption,  $P \cap B = Q$  is a nonzero ideal of  $B$  and is contained in  $L \cap B = I$ , a contradiction. The result follows.  $\square$

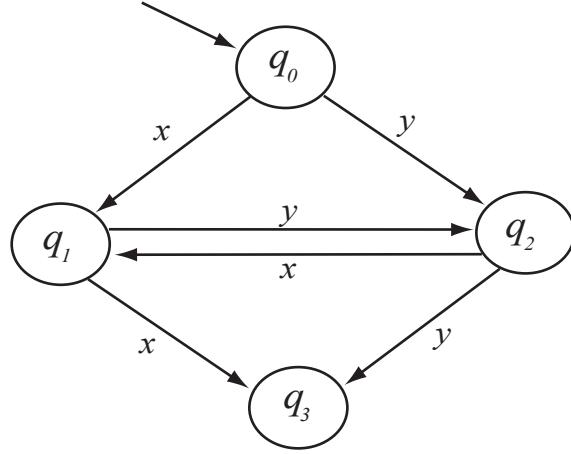


Figure 2: A finite state automaton in which the only  $q_0$ -revisiting word is the empty word.

## 4 Proofs

In this section we prove the following generalization of Theorem 1.2.

**Theorem 4.1.** *Let  $k$  be a field and let  $A$  be a prime automaton algebra over  $k$ . Then  $A$  is either primitive or  $A$  satisfies a polynomial identity.*

To prove this we need a few definitions.

**Definition.** Let  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  be a minimal finite state automaton. Given a state  $q \in Q$ , we say a word  $w \in \Sigma^*$  is  $q$ -revisiting if  $w = w'w''$  for some  $w', w'' \in \Sigma^*$  with  $w'$  non-trivial such that  $\delta(q, w') = q$ . Otherwise, we say  $w$  is  $q$ -avoiding.

A key obstruction in this proof is that there exist examples of prime automaton algebras for which there are no non-trivial words in  $\Sigma^*$  that are  $q_0$ -revisiting in the corresponding minimal automaton  $\Gamma = (Q, \Sigma, \delta, q_0, F)$ . For example, if  $A = k\{x, y\}/(x^2, y^2)$ . Then the minimal automaton corresponding to the algebra  $A$  is given in Figure 2, where the accepting states are  $F = \{q_0, q_1, q_2\}$ . We note that it is impossible to revisit the initial state  $q_0$  in this case.

Before proceeding with the generalization of Theorem 1.2, we define an equivalence relation on the accepting states of a minimal finite state automaton  $\Gamma = (Q, \Sigma, \delta, q_0, F)$ . We say that  $q_i \sim q_j$  if there exists words  $w$  and  $w'$  such that  $\delta(q_i, w) = q_j$  and  $\delta(q_j, w') = q_i$ .

Given a minimal finite state automaton  $\Gamma = (Q, \Sigma, \delta, q_0, F)$ , we can put a partial order between the equivalence classes in the following way. Let  $q$  and  $q'$  be two accepting states and let  $[q]$  and  $[q']$  denote their equivalence classes. We say that  $[q] \leq [q']$  if there is a word  $w$  such that  $\delta(q, w) = q'$ . (Note that if  $[q] \leq [q']$  and  $[q'] \leq [q]$  then  $q \sim q'$  and so the two

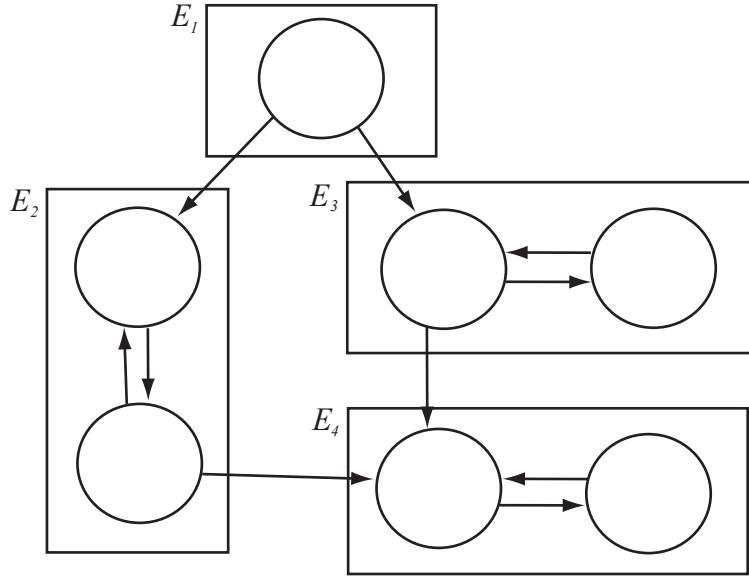


Figure 3: A finite state machine with four equivalence classes.

classes are the same.) Figure 3 gives an example of a finite state automaton in which the set of states has been partitioned into equivalence classes. In this example, the class  $E_1$  is minimal,  $E_4$  is maximal, and  $E_2$  and  $E_3$  are incomparable.

To obtain the proof of Theorem 1.2, we show that  $A$  has a well-behaved free subalgebra. We call the subalgebras we construct *state subalgebras*.

**Definition.** Let  $A$  be an automaton algebra and let  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  be its corresponding minimal finite state automaton. Given a state  $q \in F$ , we define the *state subalgebra of  $A$  corresponding to  $q$*  to be the subalgebra generated by all words  $w \in \Sigma^*$  such that  $\delta(q, w) = q$ .

**Lemma 4.2.** *Let  $A$  be an automaton algebra and let  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  be its corresponding minimal finite state automaton. A state subalgebra  $B$  of  $A$  corresponding to some state  $q$  in  $F$  is a free algebra.*

**Proof.** We claim that  $B$  is free on

$$E := \{w \in \Sigma^* \mid \delta(q, w) = q, \text{ every nonempty proper initial subword of } w \text{ is } q\text{-avoiding}\}.$$

Since  $B$  is generated by words  $w$  such that  $\delta(q, w) = q$  and every such word  $w$  can be decomposed into a product of words  $w = w_1 \cdots w_d$  with  $\delta(q, w_i) = q$  and for which every nonempty proper initial subword of  $w_i$  is  $q$ -revisiting, we see that  $B$  is generated by  $E$ . Suppose that  $B$  is not free on  $E$ . Then we have a non-trivial relation of the form

$$\sum c_{i_1, \dots, i_d} w_{i_1} \dots w_{i_d} = 0,$$

in which only finitely many of the  $c_{i_1, \dots, i_d}$  are nonzero and each  $w_{i_1}, \dots, w_{i_d} \in E$ . Since  $A$  is a monomial algebra, we infer that we must have a relation of the form

$$w_{i_1} \dots w_{i_d} = w_{j_1} \dots w_{j_e}$$

with

$$(w_{i_1}, \dots, w_{i_d}) \neq (w_{j_1}, \dots, w_{j_e}).$$

Pick such a relation with  $d$  minimal. Then note that  $w_{i_1} \neq w_{j_1}$  for otherwise, we could remove  $w_{i_1}$  from both sides and have a smaller relation:

$$w_{i_2} \dots w_{i_d} = w_{j_2} \dots w_{j_e}.$$

But then either  $w_{i_1}$  is a proper  $q$ -revisiting initial subword of  $w_{j_1}$  or vice versa, which is impossible by the definition of the set  $E$ . The result follows.  $\square$

**Lemma 4.3.** *Let  $A$  be an automaton algebra and let  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  be its corresponding minimal finite state automaton. If  $B$  is a state subalgebra of  $A$  corresponding to some state  $q$  in  $F$  then  $A$  is nearly free as a left  $B$ -module.*

**Proof.** Let

$$E = \{1\} \cup \{w \in \Sigma^* \mid \delta(q_0, w) \in F \text{ and } w \text{ is } q\text{-avoiding}\}.$$

The condition that  $\delta(q_0, w) \in F$  is just saying that  $w$  has nonzero image in  $A$ . We claim first that

$$A = \sum_{x \in E} Bx.$$

Since  $A$  is spanned by words, it is sufficient to show that every word  $w$  with nonzero image in  $A$  is of the form  $bx$  for some  $b \in B$  and  $x \in E$ . Note, however, there is some proper initial subword  $b$  of  $w$  such that  $w = bx$ ,  $\delta(q, b) = q$  and  $x$  is either  $q$ -avoiding or  $x = 1$ . Thus we obtain the first claim.

Next observe that if

$$\sum_{i=1}^d b_i x_i = 0,$$

with  $x_i \in E$ ,  $b_i \in B$  then we must have  $b_1 x_1 = \dots = b_d x_d = 0$ . To see this, observe by the argument above, every word  $u$  has a unique expression as  $bx$  for some word  $b \in B$  and  $x \in E$ . Suppose that

$$\sum_{i=1}^d b_i x_i = 0$$

and  $b_1 x_1 \neq 0$ . Then there is some word  $u$  which appears with a nonzero coefficient in  $b_1 x_1$ . But by the preceding remarks,  $u$  cannot appear with nonzero coefficient in any of  $b_2 x_2, \dots, b_d x_d$ . Since  $A$  is a monomial algebra, we obtain a contradiction. Thus  $A$  is nearly free as a left  $B$ -module.  $\square$

We have now shown that a prime automaton algebra as a free subalgebra  $B$  such that  $A$  is nearly free as a left  $B$ -module. To complete the proof that  $A$  is primitive or PI, we must show that nonzero ideals of  $A$  intersect certain state subalgebras non-trivially.

**Proposition 4.4.** *Let  $A$  be a prime automaton algebra with corresponding minimal finite state automaton  $\Gamma = (Q, \Sigma, \delta, q_0, F)$ . Suppose  $q \in F$  is in a maximal equivalence class of  $F$  under the order described above and  $B$  is the state subalgebra corresponding to  $q$ . If  $I$  is a nonzero two sided ideal of  $A$  then  $I \cap B$  is a nonzero two sided ideal of  $B$ .*

**Proof.** Every element  $x \in I$  can be written as

$$\sum c_w w,$$

where  $w \in \Sigma^*$ . Among all nonzero  $x \in I$ , pick an element

$$x = c_1 w_1 + \cdots + c_d w_d$$

with  $d$  minimal. Then we may assume  $c_1, \dots, c_d$  are all nonzero. Pick  $u$  such that  $\delta(q_0, u) = q$ . Since  $A$  is prime, there is some word  $v$  such that  $uvx \neq 0$ . Then  $uvx = c_1 u v w_1 + \cdots + c_d u v w_d$  is a nonzero element of  $I$ . By minimality of  $d$ ,  $uvw_i$  has nonzero image in  $A$  for every  $i$ . Consequently,  $\delta(q_0, uvw_i) \in F$  for all  $i$ . Since  $q$  is in a maximal equivalence class of  $F$  and  $\delta(q_0, u) = q$ ,  $\delta(q_0, uvw_i) \in [q]$  for  $1 \leq i \leq d$ .

Note that if  $\delta(q_0, uvw_i) \neq \delta(q_0, uvw_j)$  for some  $i, j$  then by minimality of  $\Gamma$ , there is some word  $w \in \Sigma^*$  such that  $\delta(q_0, uvw_i w) \in F$  and  $\delta(q_0, uvw_j w) \notin F$  (or vice versa). Consequently,  $uvw_i w$  has nonzero image in  $A$  and  $uvw_j w = 0$  in  $A$ . Thus  $uvxw$  is a nonzero element of  $I$  with a shorter expression than that of  $x$ , contradicting the minimality of  $d$ . It follows that

$$\delta(q_0, uvw_1) = \cdots = \delta(q_0, uvw_d).$$

Since  $\delta(q_0, uvw_1) \in [q]$ , there is some word  $u'$  such that  $\delta(q_0, uvw_1 u') = q$ . Consequently,

$$\delta(q_0, uvw_1 u') = \cdots = \delta(q_0, uvw_d u') = q.$$

Thus  $uvw_i u' \in B$  for  $1 \leq i \leq d$  and so  $uvxu' \in B \cap I$  is nonzero. The result follows.  $\square$

**Proof of Theorem 4.1.** It is sufficient to show that  $A$  is right primitive since the opposite ring of  $A$  is again an automaton algebra.<sup>2</sup> Let  $\Gamma = (Q, \Sigma, \delta, q_0, F)$  be the minimal finite state automaton corresponding to  $A$ . We pick a state  $q \in F$  that is in an equivalence class  $[q]$  that is maximal with respect to the order described above. We let  $B$  be the state subalgebra of  $A$  corresponding to  $q$ . By Lemma 4.2,  $B$  is a free algebra. We now have two cases.

**Case I:**  $B$  is free on at most one generator.

In this case, we claim that  $A$  satisfies a polynomial identity. Let  $u$  be a word satisfying  $\delta(q_0, u) = q$ . Let  $v$  be a word with nonzero image in  $A$ . Since  $A$  is prime, there is some word  $w$  such that  $uvw$  has nonzero image in  $A$ . Thus  $\delta(q_0, uvw) \in F$ . Since  $\delta(q_0, u) = q$ ,

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<sup>2</sup>By Kleene's theorem [1, Theorem 4.1.5], the collection of words accepted by a finite state automaton forms a regular language; by symmetry in the definition of a regular language, the *reverse* language obtained by reversing all strings in a given regular language is again regular. At the level of algebras, string reversal corresponds to multiplication of words in the opposite ring.

and  $[q]$  is a maximal equivalence class,  $\delta(q, wv) \in [q]$ . In particular, there is some word  $t$  such that  $\delta(q_0, uwvt) = q$ . Thus  $wvt \in B$ . But  $B$  is free on at most one generator. In particular every word in  $B$  must be a power of some (possibly empty word)  $b$ . Thus  $v$  is a subword of  $b^m$  for some  $m$ . It follows that every word with nonzero image in  $A$  is a subword of  $b^m$  for some  $m$ . In particular, the number of words in  $A$  of length  $n$  that have nonzero image in  $A$  is bounded by the length of  $b$ . Hence  $A$  has GK dimension at most one (cf. Krause and Lenagan [10, Chapter 1]). Thus  $A$  is PI [15].

**Case II:**  $B$  is free on two or more generators.

In this case,  $B$  is primitive by Theorem 3.1. By Lemma 4.3,  $A$  is nearly free as a left  $B$ -module. By Proposition 4.4, nonzero ideals of  $A$  intersect  $B$  non-trivially. Hence  $A$  is right primitive by Proposition 3.2. The result follows.  $\square$

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